April 11, 2018

**MATH2050C Quiz 4b**

1. (10 marks) Use $\varepsilon$-$\delta$ formulation to show that the function $f$ is continuous at $x_0$ if there is some constant $M > 0$ such that

$$|f(x) - f(x_0)| \le M|x - x_0|^2 \ , \quad \forall x \in (x_0 - 1, x_0 + 1) \ .$$

**Solution.** We let $\delta = \min\{1, \sqrt{\varepsilon/M}\}$. Then for $x, |x - x_0| < \delta$, we have $x \in (x_0 - 1, x_0 + 1)$ so

$$|f(x) - f(x_0)| \le M|x - x_0|^2 < M\delta^2 = \varepsilon \ ,$$

we conclude that $f$ is continuous at $x_0$.

2. (10 marks) Define $h(x) = 3x$ when $x$ is rational and $h(x) = 2x - 5$ when $x$ is irrational. Determine all points of continuity of $h$.

We claim the only continuity point of $h$ is $-5$. Let $x_0$ be a point of continuity of $h$. Pick a sequence of rational numbers $x_n \to x_0$, then $h(x_n) \to h(x_0)$ by continuity. As $h(x_n) = 3x_n \to 3x_0$, one must have $h(x_0) = 3x_0$. On the other hand, pick a sequence of irrational $z_n \to x_0$. We have $h(z_n) \to h(x_0)$. As $h(z_n) = 2z_n - 5 \to 2x_0 - 5$ too, one must have $h(x_0) = 2x_0 - 5$. From $3x_0 = 2x_0 - 5$, we get $x_0 = -5$.